

Wednesday, November 19, Lecture 34: Laplace Transform

Last time... Green's functions.

Given a linear operator

$$A y = f$$

like the differential eqn

$$\underbrace{\left(a \frac{d^2}{dx^2} + b \frac{d}{dx} + c \right)}_A y = f \quad \text{with boundary/initial conditions,}$$

Green's function gives inverse:

$$(I_G f)(x) = \int G(x, x') f(x') dx'$$

$$I_G = A^{-1}, \text{ so}$$

$$I_G f = y, \text{ solution}$$

We saw that the Fourier transform

$$\hat{f}(k) = F(f)(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

is in large part very useful because it turns derivatives into multiplication ("diagonalizes")

$$F\left(\frac{\partial}{\partial x} f\right)(k) = ik F(f).$$

Def (Laplace Transform)

The Laplace transform $\mathcal{L}[f]$ of a function $f: [0, \infty) \rightarrow \mathbb{C}$ is

$$\mathcal{L}[f](s) = \int_0^{\infty} f(t) e^{-st} dt, \quad s \in \mathbb{C}.$$

usually, demand $\text{Re}(s) > 0$

Compare: Fourier Transform

$$F[f](k) = \int_{-\infty}^{\infty} f(t) e^{-ikt} dt$$

still linear,

$$\mathcal{L}[af + bg] = a\mathcal{L}[f] + b\mathcal{L}[g]$$

$\text{Re}(s) > 0$

Ex:

• $f(t) = 1, \quad \mathcal{L}[f](s) = \int_0^{\infty} e^{-st} dt = \frac{1}{s}, \quad \text{Re}(s) > 0.$

• $f(t) = e^{-at}, \quad a \in \mathbb{R}, \quad \mathcal{L}[f](s) = \int_0^{\infty} e^{-at} e^{-st} dt = \frac{1}{s+a}, \quad \text{Re}(s+a) > 0.$

• $f(t) = e^{iat}, \quad a \in \mathbb{R}.$

$$\begin{aligned} \mathcal{L}[f](s) &= \int_0^{\infty} e^{iat} e^{-st} dt \\ &= \int_0^{\infty} e^{-(s-ia)t} dt \end{aligned}$$

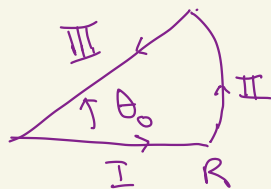
← we should take $\text{Re}(s) > 0$ to get decay.

Let $s - ia = re^{-i\theta_0}$.

Let's compute with Cauchy's thm.

$e^{-(s-ia)t}$ is h.o., so $0 = \int_{I+II+III} e^{-(s-ia)t} dt$

t is complex!

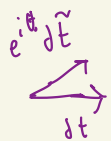


II

$$\begin{aligned}
 \left| \int_{\text{II}} e^{-(s-ia)t} dt \right| &\leq \int_0^{\theta_0} \left| e^{-re^{-i\theta_0} \cdot Re^{i\theta}} R e^{i\theta} \right| d\theta \\
 &= \int_0^{\theta_0} R \left| e^{-rR e^{i(\theta-\theta_0)}} \right| d\theta \\
 &\xrightarrow{R \rightarrow \infty} 0
 \end{aligned}$$

Exercise - Show this. (Jordan's Lemma)

$$\text{III: } \int_{\text{III}} e^{-(s-ia)t} dt = - \int_0^R e^{-re^{-i\theta_0} \tilde{t}} e^{i\theta_0} e^{i\theta_0} d\tilde{t}$$



$$= - \int_0^R e^{-r\tilde{t}} e^{i\theta_0} d\tilde{t}$$

$$R \rightarrow \infty = - \frac{e^{i\theta_0}}{r}$$

$$= - \frac{1}{re^{-i\theta_0}}$$

$$= - \frac{1}{s-ia}$$

$$\Rightarrow \boxed{\mathcal{L}[e^{iat}](s) = \frac{1}{s-ia}}$$

$$\text{Real part: } \mathcal{L}[\cos at] = \frac{s}{s^2+a^2}$$

$$\text{Im. part: } \mathcal{L}[\sin at] = \frac{a}{s^2+a^2}$$

Notice:
poles view of $\mathcal{L}[f]$
you about behavior of f !

Imaginary part w:
oscillatory behavior

$$\text{Ex: } \mathcal{L}[e^{iat}] = \frac{1}{s-ia}$$

Real part σ :
exponential decay/growth

$$\text{Ex: } \mathcal{L}[e^{-at}] = \frac{1}{s-(-a)}$$

A bit of intuition:

$$\mathcal{L}[f](\sigma + i\omega) = \int_0^{\infty} f(t) \underbrace{e^{-\sigma t}}_{\substack{\text{exponential} \\ \text{decay/growth}}} \underbrace{e^{-i\omega t}}_{\substack{\text{oscillations} \\ \text{at frequency } \omega}} dt$$

$\uparrow$ damping, resistance $\uparrow$ harmonic oscillator, AC current

Just like Fourier Transform, Laplace Tx turns derivatives into multiplication:

$$\mathcal{L}\left[\frac{d}{dt}f\right](s) = \int_0^{\infty} f'(t) e^{-st} dt$$

$$\stackrel{\text{IBP}}{=} -f(0) \cdot e^0 - \cancel{f(\infty)e^{-\infty}} + s \int_0^{\infty} f(t) e^{-st} dt$$

$$\mathcal{L}\left[\frac{d}{dt}f\right](s) = -f(0) + s\mathcal{L}[f](s) \quad \leftarrow \text{very similar to } \mathcal{F}\left(\frac{d}{dx}f\right) = ikF(f)$$

$\uparrow$ initial value

and

$$\begin{aligned} \mathcal{L}[f''](s) &= -f'(0) + s\mathcal{L}[f'] \\ &= -f'(0) + s(-f(0) + s\mathcal{L}[f]) \\ &= s^2\mathcal{L}[f] - sf(0) - f'(0). \end{aligned}$$

Lots of utility in ODEs and circuit analysis.

Ex: solve

$$y'' + 4y' + 4y = t^2 e^{-2t}, \quad y(0) = y'(0) = 0$$

$$\mathcal{L}[y''] + 4\mathcal{L}[y'] + 4\mathcal{L}[y] = \mathcal{L}[t^2 e^{-2t}]$$

$$(s^2 + 4s + 4)\mathcal{L}[y] = \frac{2}{(s+2)^3}$$

↑ often, we use table + properties. see HW.

↑
initial conditions
make stuff vanish

$$\Rightarrow \mathcal{L}[y] = \frac{2}{(s+2)^5}$$

$$\Rightarrow y = \mathcal{L}^{-1}\left[\frac{2}{(s+2)^5}\right]$$

↓ use Table online / in book

$$\frac{t^4 e^{-2t}}{12}$$

Wow! Laplace transforms turn (linear) ODEs into algebra problems.

The catch: need to compute inverse Laplace transform.

Typically, we use a table. If push comes to shove, the inverse Laplace transform is given by Bromwich integral, usually combined with Residue Thm.